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Sigma models and the ADHM construction of instantons

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Abstract

This paper is devoted to the construction of a family of linear sigma models with (0,4) supersymmetry which should flow in the infrared to the stringy version of Yang–Mills instantons on $\mathbb{R}^4$. The family depends on the full set of expected parameters and is obtained by using the data that appear in the ADHM construction of instantons.

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1. Introduction

There has long been interest in finding solutions of string theory which – in the large scale limit – go over to the familiar instanton solutions of Yang–Mills field theory. An argument using supersymmetry shows that any field theoretic instanton solution can be systematically corrected, by adding terms of higher and higher order in α', to get a string theoretic solution [1].

This perturbative approach can be made particularly explicit for those solutions that have a simple description in field theory. For instance, if the gauge group is $SU(2)$, then an instanton of topological charge k depends on $8k-3$ parameters (or $8k$ if one includes the global $SU(2)$ rotations). A sub-family depending on $5k+4$ parameters can be described by a particularly nice ansatz introduced by 't Hooft, while to describe the full $8k-3$ parameter family requires the more sophisticated ADHM construction [2]. (The ADHM construction was originally obtained using twistor space and algebraic geometry, but can be derived and explained in terms of differential geometry of $\mathbb{R}^4$ [3].) At least to first order in α', the 't Hooft ansatz fits nicely with string theory –

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it has an elegant extension [4,5] that obeys the low energy equations derived from the heterotic string. On the other hand, the special properties of the ADHM construction have not yet been exploited in any stringy way. That will be done in the present paper.

The solutions of string theory corresponding to instantons should be conformal field theories with $(0,4)$ supersymmetry, since in the field theory limit (that is the limit of a large scale instanton or equivalently $\alpha' \to 0$), self-duality of the gauge field is the condition for $(0,4)$ supersymmetry. Conformal field theories often arise as the infrared limit of theories that are not conformally invariant; a simple linear field theory can flow in the infrared to a very subtle nonlinear model. For instance, this point of view has been applied to Calabi-Yau models [6,7]. To study instantons in this way, we should consider linear sigma models with $(0,4)$ supersymmetry. The mass terms and potentials in such a model violate conformal invariance at the classical level; $(0,4)$ models with such interactions have been studied before [8], but we will here need a slightly new twist.

The conclusion we get can be stated as follows: in the right context, with the right multiplets, the condition that a $(0,1)$ model should have $(0,4)$ supersymmetry is that the Yukawa couplings should obey the ADHM equations! Thus, for every instanton - that is, every solution of the ADHM equations - we get a $(0,4)$ sigma model that should flow in the infrared to the solution of string theory corresponding to the given instanton.

This gives a uniform treatment of all instantons in string theory, and I believe that, at least to physicists, it will put the ADHM construction on more familiar grounds. [1] Moreover, it may enable one to get some information about the behavior in string theory as an instanton shrinks to zero size; we comment on this issue in Section 3. This may be helpful in understanding issues of H-monopoles and S-duality [9,10]. Also the treatment of instantons via linear sigma models probably carries over to other manifolds, such as ALE spaces, on which there is a version of the ADHM construction [11].

2. Construction of the model

2.1. Supersymmetries and multiplets

We work in two dimensional Minkowski space with coordinates τ, σ and light cone variables $\sigma^{\pm} = (\tau \pm \sigma)/\sqrt{2}$; the world-sheet metric is $ds^2 = d\tau^2 - d\sigma^2$. We are interested in models with $(0,4)$ supersymmetry. This means that there are four real right-moving supercharges. On the space of four real supercharges one could assume a symmetry group $SO(4) \cong SU(2) \times SU(2)$. However, the usual $N = 4$ superconformal algebra, to which the theories we formulate should flow in the infrared, contains only a single

[1] However, while the relation to $(0,4)$ supersymmetry makes it obvious that the gauge fields coming from the ADHM construction are self-dual, it does not necessarily shed light on the harder part of the ADHM construction, which is to show that all instantons arise from this construction. That question is simply reinterpreted as the statement that all instantons arise from linear sigma models.

$SU(2)$ factor. We will call the two $SU(2)$'s F and F'. Much of the formalism is invariant under $F \times F'$, but the eventual Lagrangians will be only F'-invariant.

We write the four supersymmetries as $Q^{AA'}$, $A, A' = 1,2$. Indices $A, B, C = 1,2$ will transform as a doublet under F, and indices $A', B', C' = 1,2$ will transform as a doublet under F'. The supercharges are real in the sense that

$$Q^{AA'} = \epsilon^{AB}\epsilon^{A'B'}Q^{\dagger}_{BB'}, \tag{2.1}$$

where ϵ_{AB} and $\epsilon_{A'B'}$ are the antisymmetric tensors of the two $SU(2)$'s. The supersymmetry algebra is to be

$$\{Q^{AA'}, Q^{BB'}\} = \epsilon^{AB}\epsilon^{A'B'}P^{+}, \tag{2.2}$$

where $P^{+} = P_{-} = -i\partial/\partial\sigma^{-}$ is the generator of translations of σ^{-}.

Consider a multiplet containing bosons and fermions related by the Q's. The multiplet usually considered in conformal field theory (assuming that the eventual Lagrangian is to be F'-invariant and not F invariant) is one in which the bosons transform as $(1/2, 0)$ under $F \times F'$.[2] We will, however, also need a multiplet in which the bosons transform as $(0, 1/2)$. We will refer to the two types of multiplets as standard and twisted multiplets, respectively. Other multiplets have been found recently [12].

First, we write down the structure of the standard multiplet. (We will work in components, as I do not know how to describe these models in superspace.) There is a bose field X^{AY}, $A, Y = 1,2$, with a reality condition analogous to (2.1):

$$X^{AY} = \epsilon^{AB}\epsilon^{YZ}\overline{X}_{BZ}. \tag{2.3}$$

This multiplet admits the action of yet another $SU(2)$ group H (which will generally not be a symmetry of the theories that we eventually consider). F acts on indices A, B, C and H on the indices Y, Z; the X's thus transform as $(1/2, 0, 1/2)$ under $F \times F' \times H$. X is related by supersymmetry to a right-moving[3] fermi field $\psi_{-}^{A'Y}$ with a reality condition of the same form as (2.3). The supersymmetry transformation laws are

$$\delta X^{AY} = i\epsilon_{A'B'}\eta_{+}^{AA'}\psi_{-}^{B'Y}, \qquad \delta\psi^{A'Y} = \epsilon_{AB}\eta_{+}^{AA'}\partial_{-}X^{BY}, \tag{2.4}$$

where $\eta_{+}^{AA'}$ are infinitesimal parameters. It is easy to check the commutation relations

$$[\delta_{\eta'}, \delta_{\eta}] = -i\epsilon_{AB}\epsilon_{A'B'}\eta_{+}^{AA'}\eta'^{BB'}_{+}\partial_{-}. \tag{2.5}$$

If one wishes to consider k such multiplets, one simply extends the Y index to run from $1, \ldots, 2k$; the group H becomes the symplectic group $Sp(k)$ instead of $SU(2) \cong Sp(1)$, and the tensor ϵ^{YZ} in (2.3) should be understood as the invariant antisymmetric tensor of $Sp(k)$.

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[2] The reason for this is that, in a conformal $(0,4)$ theory, if F' is the only global symmetry, as is usually the case, it must be carried only by right-moving modes, but the bosons have also a left-moving part. In the models we consider below, the bosons that are not F'-invariant are massive and irrelevant in the infrared.

[3] ψ_{-} is right-moving in the sense that the equation of motion of the free ψ_{-} field would be $\partial_{+}\psi_{-} = 0$, so in the free theory ψ_{-} is right-moving. A similar statement holds for what we will later call left-moving fields.

The structure of the twisted multiplet is the same, except that the roles of F and F' are reversed. Thus, there is a bose field $\phi^{A'Y'}$, $A', Y' = 1, 2$ with a reality condition like that in (2.3):

$$\phi^{A'Y'} = \epsilon^{A'B'}\epsilon^{Y'Z'}\overline{\phi}_{B'Z'}. \tag{2.6}$$

The A' index transforms as a doublet of F', and Y' is acted on by a new $SU(2)$ group H'. In addition, there is a right-moving fermi multiplet $\chi_-^{AY'}$ (an F' singlet, as the notation indicates), obeying a reality condition like that in (2.6). The transformation laws are obtained from (2.4) with obvious substitutions:

$$\delta\phi^{A'Y'} = i\epsilon_{AB}\eta_+^{AA'}\chi_-^{BY'}, \qquad \delta\chi_-^{AY'} = \epsilon_{A'B'}\eta_+^{AA'}\partial_-\phi^{B'Y'}. \tag{2.7}$$

The commutation relations of (2.5) are of course obeyed. For k' such multiplets, one must take H' to be $Sp(k')$, take Y' to run from $1, \dots, 2k'$ and interpret the tensor $\epsilon^{Y'Z'}$ in the reality condition as the invariant antisymmetric tensor of $Sp(k')$.

Of course, until we start writing Lagrangians, there is no essential difference between F and F', and the two multiplets are on the same footing. They become distinguished once we construct Lagrangians that are invariant under F' and not F.

2.2. *Left-moving fermions*

Now we introduce left-moving fermions λ_+^a, $a = 1, \dots, n$. The part of the Lagrangian containing λ_+^a is of the general form

$$\int d^2\sigma \left(\frac{i}{2}\lambda_+^a\partial_-\lambda_+^a - \frac{i}{2}\lambda_+^a G_{a\theta}\rho_-^\theta\right), \tag{2.8}$$

where the components of ρ_- include all the right-moving fermions ψ, χ, and $G_{a\theta}$ is some unknown function of X and ϕ incorporating masses and Yukawa couplings. The equations of motion of the λ's are thus

$$\partial_-\lambda_+^a = G_\theta^a\rho_-^\theta. \tag{2.9}$$

Since I do not know how to treat the λ's as a $(0,4)$ superfield, we will have to study the supersymmetry transformations on-shell and by hand. The most general supersymmetry transformation allowed for the λ's by dimensional analysis is

$$\delta\lambda_+^a = \eta_+^{AA'}C_{AA'}^a, \tag{2.10}$$

with C a function of X, ϕ. Calculating the second variation, we get

$$\delta_{\eta'}\delta_\eta\lambda_+^a = i\eta_+^{AA'}\left(\frac{\partial C_{AA'}^a}{\partial X^{BY}}\epsilon_{B'C'}\eta'^{BB'}_+\psi_-^{C'Y} + \frac{\partial C_{AA'}^a}{\partial\phi^{B'Y'}}\epsilon_{BC}\eta'^{BB'}_+\chi^{CY'}\right). \tag{2.11}$$

We want to compare this to the expected result

$$\begin{aligned}\left(\delta_{\eta'}\delta_\eta - \delta_\eta\delta_{\eta'}\right)\lambda_+^a &= -i\epsilon_{A'B'}\epsilon_{AB}\eta_+^{AA'}\eta'^{BB'}_+\partial_-\lambda_+^a \\ &= -i\epsilon_{AB}\epsilon_{A'B'}\eta_+^{AA'}\eta'^{BB'}_+G_\theta^a\rho^\theta.\end{aligned} \tag{2.12}$$

The condition for (2.11) to be of the form (2.12) – for a suitable G – is simply that

$$0=\frac{\partial C^a_{AA'}}{\partial X^{BY}}+\frac{\partial C^a_{BA'}}{\partial X^{AY}}=\frac{\partial C^a_{AA'}}{\partial \phi^{B'Y'}}+\frac{\partial C^a_{AB'}}{\partial \phi^{A'Y'}}. \tag{2.13}$$

If this is so, then (2.11) and (2.12) agree with

$$G^a_\theta \rho^\theta=\frac{1}{2}\left(\epsilon^{BD}\frac{\partial C^a_{BB'}}{\partial X^{DY}}\psi_-^{B'Y}+\epsilon^{B'D'}\frac{\partial C^a_{BB'}}{\partial \phi^{D'Y'}}\chi^{BY'}\right). \tag{2.14}$$

In (2.10) and (2.13), we have obtained a considerable amount of information about the supersymmetry transformation law of λ. But is there actually a Lagrangian that is invariant under this symmetry? To answer this question, it is useful to compare to $(0,1)$ supersymmetry, where there is a superspace formulation. In that case, λ is part of a multiplet $\Lambda^a=\lambda^a+\theta F^a$, where F is an auxiliary field that eventually (by equations of motion) becomes a function of the other bosons. The supersymmetry transformation law of λ is

$$\delta\lambda^a=\eta F^a \tag{2.15}$$

and the potential energy of the theory is

$$V=\frac{1}{2}\sum_a F^a F^a. \tag{2.16}$$

Comparing (2.15) to (2.10), it is clear that F corresponds to a component of C. If we pick any real c-number $c^{AA'}$, normalized so

$$\epsilon_{AB}\epsilon_{A'B'}c^{AA'}c^{BB'}=1, \tag{2.17}$$

then we can make a $(0,1)$ supersymmetric model in which the transformation laws of λ are the ones given above specialized to $\eta^{AA'}=\eta c^{AA'}$, η being an anticommuting parameter of $(0,1)$ supersymmetry. The potential of this theory will be

$$V=\frac{1}{2}\sum_a C^a C^a, \tag{2.18}$$

where

$$C^a=c^{AA'}C^a_{AA'}. \tag{2.19}$$

To get $(0,1)$ supersymmetry with the transformation laws in (2.10), we need the potential in (2.18) to be independent of c, ensuring that the *same* Lagrangian is invariant under all four supersymmetries. The condition for this is that

$$0=\sum_a\left(C^a_{AA'}C^a_{BB'}+C^a_{BA'}C^a_{AB'}\right). \tag{2.20}$$

We now have enough information to determine what the $(0,4)$ Lagrangian is, if there is one. The Yukawa couplings are given in (2.8) and the potential in (2.18). The Lagrangian is therefore

$$L = L_{\rm kin} - \frac{i}{4}\int d^2\sigma\lambda_+^a\left(\epsilon^{BD}\frac{\partial C^a_{BB'}}{\partial X^{DY}}\psi_-^{B'Y} + \epsilon^{B'D'}\frac{\partial C^a_{BB'}}{\partial\phi^{D'Y'}}\chi_-^{BY'}\right)$$
$$- \frac{1}{8}\int d^2\sigma\epsilon^{AB}\epsilon^{A'B'}C^a_{AA'}C^a_{BB'}, \qquad (2.21)$$

where $L_{\rm kin}$ is the free kinetic energy for all fields. $(0,4)$ supersymmetry does hold when (2.13) and (2.20) are valid since (2.20) insures that the Lagrangian is invariant under all the supersymmetries and (2.13) ensures that these generate the global $(0,4)$ algebra.

2.3. The ADHM equations

Now let us make the meaning of (2.13) and (2.20) more explicit. (2.13) can be solved quite explicitly; the general solution (raising and lowering indices with the ϵ symbols) is

$$C^a_{AA'} = M^a_{AA'} + X_{AY}N^{aY}_{A'} + \phi_{A'}{}^{Y'}D^a_{AY'} + X_A{}^Y\phi_{A'}{}^{Y'}E^a_{YY'}, \qquad (2.22)$$

where M, N, D, and E are independent of X and ϕ. Since C thus contains only terms at most bilinear in X and ϕ, the potential $V \sim C^2$ at most contains terms of degree $(2,2)$.

It is straightforward now to impose Eqs. (2.20), giving a finite system of equations for the finite set of coefficients in M, N, D, E. However, to imitate as much as possible the structure of superconformal $(0,4)$ theories, we want to impose invariance under F' (or F). F' invariance simply means that $M = N = 0$, since $M^a_{AA'}$ and $N^a_{A'Y}$ transform as doublets of F' (which acts on the A' subscript). Thus in the F' invariant case, we can succinctly write

$$C^a_{AA'} = B^a_{AY'}\phi_{A'}{}^{Y'}, \qquad (2.23)$$

where B is linear in X (and independent of ϕ). Eq. (2.20) can then be conveniently written

$$\sum_a \left(B^a_{AY'}B^a_{BZ'} + B^a_{BY'}B^a_{AZ'}\right) = 0. \qquad (2.24)$$

Thus, the F'-invariant $(0,4)$ theories are simply determined by a tensor B, linear in X, and obeying (2.24). Of course, it may also be interesting to study the theories that arise if one does not assume either F or F' invariance.

Now let us discuss the physical significance of this model. Since C is homogeneous and linear in ϕ, the potential V is homogeneous and quadratic in ϕ; in particular, it vanishes at $\phi = 0$, for any X. Therefore, the X's are massless fields, but the ϕ's are massive; indeed, being quadratic in ϕ, V can be interpreted as an X-dependent mass term for the ϕ's. If the number n of components of λ is big enough and the solution B of (2.24) is sufficiently generic, then for every value of X, all of the ϕ's are massive. (From what we will presently say, that this generically happens follows from standard theorems about the ADHM construction.) So the massless particles are precisely the X's, and the space of vacua is $\mathcal{M} = \mathbb{R}^{4k}$, parametrized by the X's. (We recall that k and

k' are the numbers of X and ϕ multiplets.) At the classical level, the metric on $\mathcal{M}$ is read off from the classical Lagrangian after setting the ϕ's to zero, and (assuming that we started with the free kinetic energy for all fields), it is just the flat metric on $\mathbb{R}^{4k}$. Of course, the $(0,4)$ supersymmetry endows $\mathcal{M}$ with a hyper-Kähler structure.

What about the fermions? On $\mathcal{M}$, that is, at $\phi = 0$, the structure of the Yukawa couplings is particularly simple; it reduces to

$$\sum_a \lambda_+^a B^a_{AY'} \chi_-^{AY'} . \tag{2.25}$$

Thus, the fermionic partners ψ_- of X are all massless, as one would expect from $(0,4)$ supersymmetry. If the number n of components of λ_+^a is bigger than the number $4k'$ of components of $\chi_-^{AY'}$, then generically all components of χ_- get mass (in fact, by supersymmetry this is true precisely when all components of ϕ are massive), and $N = n - 4k'$ components of λ_+ are massless.

Let $v_i{}^a$, $i = 1,\dots,N$, be a basis of the massless components of λ_+, that is the solutions of $\sum_a v_i^a B^a_{AY'} = 0$. Choose the v's to be orthonormal, that is $\sum_a v_i{}^a v_j{}^a = \delta_{ij}$. Of course, the v_i^a are X-dependent, as the tensor B is X-dependent (in fact, linear in X). Orthonormality determines v_i^a up to an X-dependent $SO(N)$ transformation on the i index; this will be interpreted presently as the gauge invariance of the low energy theory. One can set to zero the massive modes in λ_+ by writing

$$\lambda_+^a = \sum_{i=1}^{N} v_i^a \lambda_{+\,i}, \tag{2.26}$$

where $\lambda_{+\,i}$ are the massless left-moving fermions.

To write, at the classical level, the effective action for the massless modes, one sets to zero the massive fields ϕ, χ, and makes the ansatz (2.26) for λ_+. In particular, the kinetic energy for λ_+ becomes

$$\begin{aligned} &\frac{i}{2}\int d^2\sigma \sum_{i,j,a} \left(v_i^a \lambda_{+\,i}\right) \partial_- \left(v_j^a \lambda_{+\,j}\right) \\ &\quad = \frac{i}{2}\int d^2\sigma \sum_{i,j} \left\{ \lambda_{+\,i} \left(\delta_{ij}\partial_- + \partial_- X^{BB'} A_{BB'\,ij}\right) \lambda_{+\,j} \right\}, \end{aligned} \tag{2.27}$$

with

$$A_{BB'\,ij} = \sum_a v_i^a \frac{\partial v_j^a}{\partial X^{BB'}} . \tag{2.28}$$

(2.27) is the standard sigma model expression for couplings of left-moving fermions to space–time gauge fields, the gauge field being given in (2.28).

$(0,4)$ supersymmetry means that the gauge field in (2.28) must be compatible with the hyper-Kähler structure of $\mathbb{R}^{4k}$ (that is, its curvature is of type $(1,1)$ for each of the complex structures). For $k = 1$, this reduces to a more familiar statement: the gauge field is an instanton; its curvature is anti-self-dual.

In fact, for $k = 1$, what we have just obtained is simply the ADHM construction of instantons. According to the ADHM construction, an instanton of $SO(N)$, with instanton number k', is given by a tensor B, linear in X, that obeys (2.24), and has a further non-degeneracy condition which simply asserts that the components of ϕ are all massive for all X. Moreover, (2.28) is the standard ADHM formula for the gauge field, in terms of B. The definition of the gauge field is actually perhaps better explained without formulas; the bundle E of massless fermions is a subbundle of the trivial bundle E_0 of all fermions λ_+^a, and the gauge connection on E is the connection on E induced from the trivial connection on E_0. In formulas, that corresponds to (2.28).

The ADHM theorem further asserts that two instantons are equivalent if and only if the two B tensors can be mapped into each other by the action of $SO(n) \times Sp(k')$ (which comes from the linear action of this group on the a and Y' indices of $\lambda_+{}^a$ and $\phi^{A'Y'}$).

The ADHM construction gives rise to a partial compactification of instanton moduli space in which one drops the non-degeneracy condition and permits instantons for which ϕ is massless at some values of X; this corresponds to allowing some instantons to shrink to zero size. One wonders if this compactification is relevant to string theory, especially since linear sigma models have a good record of predicting correctly the moduli spaces of conformal field theories including the subtle behavior associated with singularities (for instance, see [13]). The partial compactification of moduli space that comes from the ADHM construction is actually the analog for $\mathbb{R}^4$ of the compactification that has been (rather optimistically) used in testing S-duality of $N = 4$ super Yang-Mills theory for strong coupling [10].

3. Instanton number one

In this section, I will make this construction somewhat more explicit by describing precisely the instanton number one solution in this language. Of course, there is no essential novelty here; the formulas are well known in the ADHM literature. Writing them out here may nevertheless be useful.

The basic one instanton solution arises for gauge group $SU(2)$, but of course can be embedded in any larger gauge group. In the formulation above, the natural gauge group is $SO(N)$. We will take $N = 4$ and use the embedding of $SU(2)$ in $SO(4)$ given by the decomposition $SO(4) \cong SU(2) \times SU(2)$. Thus, we identify the gauge group of the one instanton solution with one of the $SU(2)$'s in $SO(4)$; the second $SU(2)$ – call it K – is a (rather trivial) global symmetry group of the solution.

Now, let us consider the full symmetry group of the sigma model corresponding to this solution. Any $(0,4)$ sigma model of the type we are considering has a global symmetry group $F' \cong SU(2)$. In addition, the one instanton solution happens to be invariant under rotations[4] of $\mathbb{R}^4$. The rotation group is $SO(4) \cong SU(2)_L \times SU(2)_R$. So altogether, the global symmetry of the (0,4) sigma model that we are looking for is $F' \times K \times SU(2)_L \times SU(2)_R \cong SU(2)^4$.

What are the fields supposed to be? We want $k = 1$ to get $\mathbb{R}^4$ as space-time, and $k' = 1$ so that the instanton number will be one. There are therefore four X's, X^{AY}, and four ϕ's, $\phi^{A'Y'}$. On the X's and ϕ's there is a natural action, noted in the last section, of $F \times F' \times H \times H' \cong SU(2)^4$. It is tempting to try to identify $F \times F' \times H \times H'$ with the isomorphic group $F' \times K \times SU(2)_L \times SU(2)_R$, and this indeed proves to be correct. Thus, while the general instanton-related (0,4) model of the previous section has only F' symmetry, the one instanton solution (embedded as we have described in $SO(4)$) proves to have the full $F \times F' \times H \times H'$ symmetry.

To achieve this symmetry in the Lagrangian, one needs an appropriate action of $F \times F' \times H \times H'$ on the left-moving fermions λ_+. The number n of λ_+ components is 8 (so that $N = n - 4k' = 4$), and with a little experimentation (or by comparing to the standard ADHM description of the basic instanton) one finds that the λ_+'s must be taken to transform as $(1/2,0,0,1/2) \oplus (0,0,1/2,1/2)$. The λ_+ fields are thus naturally written as $\lambda_+^{AY'}$, $A, Y' = 1,2$ and $\lambda_+^{YY'}$, $Y, Y' = 1,2$, with the usual sort of reality condition

$$\lambda_+^{AY'} = \epsilon^{AB}\epsilon^{Y'Z'}\bar{\lambda}_{+BZ'}, \qquad \lambda_+^{YY'} = \epsilon^{YZ}\epsilon^{Y'Z'}\bar{\lambda}_{+ZZ'}. \tag{3.1}$$

Note that the total number of components of λ_+ is indeed $2 \times 2 + 2 \times 2 = 8$.

For this choice of λ_+'s, the coupling tensor $C^a_{BB'}$ has two pieces, $C^{AY'}{}_{BB'}$ and $C^{YY'}{}_{BB'}$. For these we make (up to inessential rescalings) the most general ansatz compatible with (2.22) and the symmetries:

$$C^{YY'}{}_{BB'} = X_B{}^Y \phi_{B'}{}^{Y'}, \qquad C^{AY'}{}_{BB'} = \frac{\rho}{\sqrt{2}}\delta^A{}_B \phi_{B'}{}^{Y'}. \tag{3.2}$$

Here ρ is a real number that will be interpreted as the instanton scale parameter. It is easy to verify the ADHM equations (2.24). One may readily compute from (2.21) that the potential is

$$V = \tfrac{1}{8}(X^2 + \rho^2)\phi^2, \tag{3.3}$$

where

$$X^2 = \epsilon_{AB}\epsilon_{YZ}X^{AY}X^{BZ}, \qquad \phi^2 = \epsilon_{A'B'}\epsilon_{Y'Z'}\phi^{A'Y'}\phi^{B'Z'}. \tag{3.4}$$

The expected symmetry under $F \times F' \times H \times H'$, which acts by independent rotations of X and ϕ, is manifest in (3.3).

To extract the standard formula for the one instanton solution from these expressions, we simply need to write down the low energy action for the massless fermions. The following ansatz exhibits four massless modes of λ_+:

[4] In field theory, the instanton solution is actually invariant in addition under certain conformal transformations of $\mathbb{R}^4$, but this depends on the conformal invariance of the self-dual Yang-Mills equations and is lost in string theory.

$$\lambda^{YY'} = \frac{\rho\zeta_+^{YY'}}{\sqrt{\rho^2+X^2}}, \qquad \lambda^{AY'} = -\frac{\sqrt{2}X^A{}_Y\zeta_+^{YY'}}{\sqrt{\rho^2+X^2}}. \tag{3.5}$$

With this ansatz, the free kinetic energy of the λ_+'s (the first term in (2.8)) can be rewritten using the following formula:

$$\begin{aligned}&\lambda_{+\,YY'}\partial_-\lambda_+^{YY'} + \lambda_{+\,AY'}\partial_-\lambda_+^{AY'}\\ &= \zeta_{+\,YY'}\partial_-\zeta_+^{YY'} - \zeta_{YY'}\frac{\frac{1}{2}\epsilon_{AB}\left(X^{AY}\partial_-X^{BZ}+X^{AZ}\partial_-X^{BY}\right)}{X^2+\rho^2}\zeta_{+Z}{}^{Y'}.\end{aligned} \tag{3.6}$$

In the last term one sees the standard instanton gauge field.

This completes what I will say about the classical sigma model describing the instanton. To obtain the stringy corrections to the solution, one must consider the renormalization group flow to a (presumed) $(0,4)$ conformal field theory in the infrared. This certainly entails integrating out the massive fields rather than simply setting them to zero. Whether in addition it is necessary to consider renormalization group flow of the massless fields depends on whether the $(0,4)$ action that one gets by integrating out the massive fields is automatically conformally invariant. It has been argued that massless $(0,4)$ theories of the general type under discussion here are automatically conformally invariant under some conditions [14], but there are limitations on this argument [14,5]. I will only make two observations here about the renormalization group flow:

(1) In integrating out the massive fields, the expansion parameter is easily seen to be $1/(X^2+\rho^2)$; thus perturbation theory is accurate for all X if ρ is large, and for large X even if ρ is small.

(2) Despite being super-renormalizable, it appears that when formulated on a curved two-manifold, the theory has a one-loop logarithmic divergence proportional to the world-sheet curvature R, if minimally coupled to the world-sheet metric. It is conceivable that to avoid this, one should add an extra coupling, perhaps of the form $R\phi^2$, when working on a curved world-sheet. This would spoil the $\rho = 0$ symmetry between X and ϕ that is mentioned below.

3.1. *Behavior for $\rho = 0$*

It is interesting to ask what happens if ρ is small – in fact, for $\rho \to 0$. This is the regime in which the deviations of the stringy instanton from the field theoretic instanton should be large. In [4,5], an interesting proposal was made for the structure of the solution at $\rho = 0$: the low energy string-derived field equations were solved for arbitrary ρ, and it was seen that for $\rho = 0$ the solution develops a semi-infinite tube that joins to the rest of space-time near $X = 0$. This picture is not guaranteed to be correct because it is based on solving low energy equations that are not valid near $X = 0$ when $\rho = 0$. [5] From the mean field theory developed in the present paper, one gets a somewhat different

[5] The picture is supported by the fact, noted in [5], that the semi-infinite tube does indeed correspond to an exact conformal field theory. (This is true for $(0,4)$ as well as for the $(4,4)$ case discussed more thoroughly in [5].) However, one does not know if the tube is stable.

picture of what happens at $\rho = 0$. (This picture is also not guaranteed to be correct since we do not have precise control on the renormalization group flow.)

First of all, at $\rho = 0$, the massless fermions, in view of the above formulas, are simply the $\lambda_+^{AY'}$; they are exactly decoupled from the other fields, so the gauge connection is zero. This statement agrees with the picture of [4,5] concerning what happens to the gauge fields for $\rho = 0$. The space–time looks somewhat different, however. The structure of the potential energy

$$V = \frac{1}{8}(X^2 + \rho^2)\phi^2 \tag{3.7}$$

indicates that what happens at $\rho = 0$ is that the theory develops a second branch of classical vacua: in addition to the usual branch, $\phi = 0$ with any X, one has a second branch, $X = 0$ with any ϕ. Field theory is a good approximation for large X on the first branch, or for large ϕ on the second branch. Indeed, at $\rho = 0$ there is a $\mathbb{Z}_2$ symmetry that exchanges X and ϕ (unless this is ruined by an $R\phi^2$ term mentioned above). The two branches are two asymptotically flat and empty space–times, connected by a "worm-hole" region near $X = \phi = 0$ where stringy effects are important.

Or alternatively, it may be that under the renormalization group flow to the infrared, the two branches become disconnected. If in renormalization group flow the two branches become infinitely far apart, each separated from $X = \phi = 0$ by a semi-infinite tube, then it might be that the picture of [4,5] corresponds to the X branch. In any event, the meaning of the ϕ branch is rather mysterious.

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